Spherical Tiling by Congruent Pentagons

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webpage for further reading:
http://www.math.ust.hk/~mamyan/research/UROP.shtml

We consider tilings of the sphere by congruent pentagons. The basic example is dodecahedron, which has 12 pentagonal tiles.

To simplify the problem, we assume the tiling is edge-to-edge and all vertices have degree ≥ 3.

1 Numerical

Let v, e, f be the numbers of vertices, edges and faces in a spherical pentagonal tiling. Then we have

$$v - e + f = 2, \quad 5f = 2e.$$

Let v_i be the number of vertices of degree i. Then

$$v = v_3 + v_4 + v_5 + v_6 + \cdots, \quad 2e = 3v_3 + 4v_4 + 5v_5 + 6v_6 + \cdots.$$

It then easily follows that

$$v_3 = 20 + 2v_4 + 5v_5 + 8v_6 + \cdots, \quad \frac{f}{2} - 6 = v_4 + 2v_5 + 3v_6 + \cdots.$$

Exercise 1. Derive similar equalities for tiling of the sphere by quadrilaterals. What about triangles?

We conclude that f must be an even number ≥ 12. Moreover, vast majority of vertices have degree 3. We call vertices of degree ≥ 4 *high degree* vertices.

Theorem 1. *A tiling cannot have only one high degree vertices. If the tiling has exactly two high degree vertices, then the tiling is one of five families of* earth map tilings*.*

The theorem [6] does not make use of edge lengths and angles, and concerns only with the *topological structure* of the tiling.

Exercise 2. Show that the following are equivalent for a pentagonal spherical tiling.

1. The number of tiles is $f = 12$.
2. The number of vertices is $v = 20$.
3. All vertices have degree 3.
4. The tiling is the dodecahedron.

Theorem 2. *In any pentagonal tiling, there must be one tile with four vertices having degree* 3*, and the fifth vertex having degree* 3*,* 4 *or* 5*.*

The theorem [2, Proposition 1] provides a starting point finding all the tilings of the sphere by congruent pentagons. The idea is that a piece of the tiling is given by a special tile in the theorem, with 5, 6 or 7 more tiles around it. These tiles form three possible neighborhood tilings. We may first to put edge lengths and angles to such neighborhoods, and then try to tile beyond the neighborhoods.

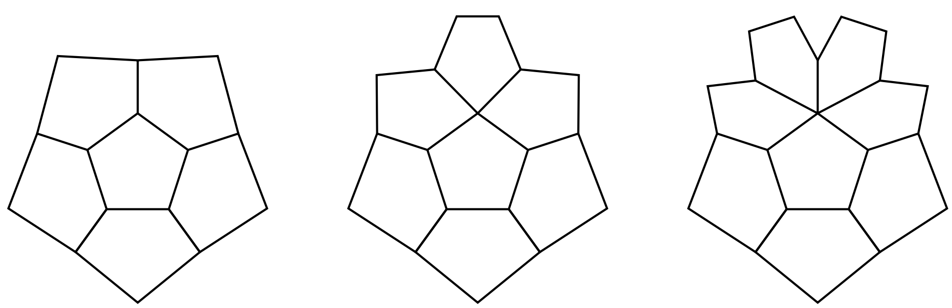

Figure 1: Three neighborhood tilings.

Exercise 3. If there is no special tile as described in Theorem 2, then every tile must be one of the following two types

1. It has at least one vertex of degree ≥ 6.

2. It has two vertices of degree 4 or 5.

Using the fact that a degree k vertex is shared by k tiles, prove the the theorem.

Exercise 4. Show that there is no similar theorem for quadrilaterals.

Problem 1. Finding all tilings with few (say 3) high degree vertices. The next case is 3 high degree vertices of distance 3 from each other.

Problem 2. Finding all tilings such that high degree vertices are "evenly distributed". For example, each tile has exactly one high degree vertex. I have a construction for such tilings, and I believe my construction gives all such tilings.

Problem 3. Similar study for quadrilateral tilings.

2 Neighborhood Tiling and Beyond

As suggested by Theorem 2, we may consider putting edges and angles into three possible neighborhood tilings, such that all pentagons are congruent.

We will concentrate on the first type, in which a center tile has all vertices having degree 3. The neighborhood of the tile consists of 6 pentagons. We first ignore the angles and try to achieve *egde congruence*.

Proposition 3. *If a spherical tiling by (edge) congruent pentagons has a tile with all vertices having degree* 3*, then the edges of the pentagon must be one of the five kinds:* $a^5, a^4b, a^3b^2, a^3bc, a^2b^2c$*.*

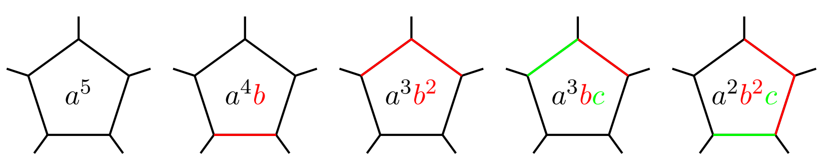

Figure 2: Five possible edge combos for the first neighborhood in Figure 1.

Exercise 5. Use Theorem 2 to prove that it is impossible for the pentagon in a spherical tiling by congruent pentagons to have all five edges having distinct edge lengths.

For the minimal case of $f = 12$ (i.e., dodecahedron), it is possible to further find all the edge congruent tilings of the sphere. The numbers of edge congruent tilings of the dodecahedron sphere are 1, 5, 1, 0, 1 for the five cases [3, Propositions 9, 10, 11, 12].

Independent of edge lengths, it is also possible to further find all the angle congruent tilings of the sphere for $f = 12$. Specifically, we first find all the possible angle combinations at vertices, which we call *anglewise vertex combination*, or AVC for short. There are five such possible AVCs [3, Propositions 17]. Then for each AVC, we may further find edge congruent tilings of the sphere [3, Propositions 18, 19, 20].

The results on edge and angle congruent tilings can then be combined together and gives the complete classification of dodecahedron tilings.

Theorem 4. *There is exactly one family of tilings of the sphere by* 12 *congruent pentagons.*

Problem 4. For the second AVC (for $f = 12$), there are many possible angle congruent tilings. Some examples are given in [3]. Can you find all the possible angle congruent tilings?

Problem 5. The family of dodecahedron tilings allows two free parameters. Find the exact range of the free parameters.

From now on, we may assume $f > 12$, which really means that f is an even number ≥ 16. It turns out that the tilings of the first neighborhood in Figure 1 with the edge combinations a^3bc and a^2b^2c leads to $f = 12$. So the cases can be dismissed.

For the edge combination a^3b^2, there are four possible tilings of the neighborhood [2, Proposition 2]. Moreover, we have a very good idea about the possible AVCs. The AVC helps us to further construct the tiling beyond the neighborhood. It turns out all four neighborhood tilings lead to contradiction.

Theorem 5. *If a spherical tiling by more than* 12 *geometrically congruent pentagons has edge length combination* a^2b^2c, a^3bc, *or* a^3b^2, *with* a, b, c *distinct, then every tile has at least one vertex of degree* > 3.

Exercise 6. Show that the neighborhood tiling with edge length a^2b^2c must be given by Figure 2. Then fill in the angles to get all pentagons to become congruent. Finally, use the angle sum equation for the pentagon (4π is the area of the sphere)

$$\sum(\text{five angles in pentagon}) - 3\pi = \text{Area(pentagon)} = \frac{4\pi}{f}. \tag{2.1}$$

to prove $f = 12$.

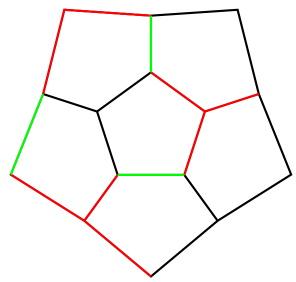

Figure 3: Edge congruent neighborhood tilings for a^2b^2c.

Problem 6. The discussion so far assumes that the whole tiling contains a part like the first neighborhood in Figure 1. We still need to consider the other two neighborhoods. What are the edge congruent tilings of the other two neighborhoods? For each such edge congruent tilings, how can you further fill in the angles so that all pentagons are congruent?

3 Equilateral Pentagonal Tiling

The technique of the last section relies heavily on the variations in the edge length. For the other extreme that all edges have equal length (i.e., equilateral pentagons), the technique is useless, and completely new strategy is needed.

Note that general pentagons allow 7 free parameters. The equilateral condition introduces 4 equalities among these parameters. Therefore equilateral pentagons allow $7 - 4 = 3$ free parameters. This means that, if we can find another 3 independent equalities among the parameters, then the equilateral pentagon is completely determined. For a specific pentagon, it is then quite easy to construct the tiling (or more likely, to show the pentagon cannot tile the sphere).

The 3 equalities can be found by the fact that the sum of all angles at a vertex must be 2π. The following gives all the possible AVCs [1, Lemma 1].

Proposition 6. *If an edge-to-edge tiling of a surface has at most five distinct angles at degree* 3 *vertices, then after suitable relabeling of the distinct angles, the anglewise vertex combination at degree* 3 *vertices is in Table 1.*

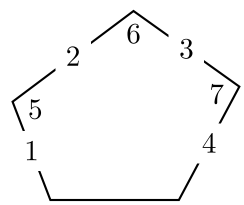

Figure 4: Number of freedoms in a pentagon.

The necessary combinations in the table must appear, and the optional ones may or may not appear. Further counting (i.e., "statistical") argument and geometrical reason show that most of the optional ones must also appear [1, Lemmas 4-7]. In fact, with the only exception of $\{\alpha\beta\gamma, \delta\epsilon^2\}$, all the AVCs gives three equalities among the five angles. Then a massive numerical calculation of the spherical pentagons can be carried out. Many pentagons are dismissed from tiling due to two reasons

1. Violation of geometric constraint [1, Lemma 7].
2. The number f of tiles as calculated by (2.1) is not an even number ≥ 16.

At the end, we get the following.

Theorem 7. *There are exactly* 8 *tilings of the sphere by equilateral pentagons.*

4 Almost Equilateral Pentagonal Tiling

The technique discussed so far should be sufficient for the case of enough variation in edge length and the case of all edges having the same length (although much remains to be done). The most difficult case is the edge length combination a^4b. There is barely enough variation in edge length, and yet there is one more freedom so that the pentagon cannot be completely determined by numerical calculation. Some progress has been made in the simplest case of the neighborhood in Figure 1, and there are at most three distinct angles in the pentagon.

<table>
<tr><th colspan="3">Necessary</th><th>Optional</th></tr>
<tr><td>α^3</td><td colspan="2"></td><td></td></tr>
<tr><td>$\alpha\beta^2$</td><td colspan="2"></td><td></td></tr>
<tr><td colspan="2">$\alpha\beta\gamma$</td><td rowspan="3"></td><td>α^3</td></tr>
<tr><td rowspan="2">$\alpha\beta^2$</td><td>$\alpha^2\gamma$</td><td rowspan="2"></td></tr>
<tr><td>γ^3</td></tr>
<tr><td colspan="2" rowspan="5">$\alpha\beta\gamma$</td><td rowspan="2">$\alpha\delta^2$</td><td>$\beta^2\delta$</td></tr>
<tr><td>β^3</td></tr>
<tr><td rowspan="2">$\alpha^2\delta$</td><td>$\beta\delta^2$</td></tr>
<tr><td>β^3</td></tr>
<tr><td>δ^3</td><td></td></tr>
<tr><td rowspan="2">$\alpha\beta^2$</td><td colspan="2">$\gamma\delta^2$</td><td>$\alpha^2\delta$</td></tr>
<tr><td colspan="2">$\alpha^2\gamma, \delta^3$</td><td></td></tr>
<tr><td rowspan="5">$\alpha\beta\gamma$</td><td colspan="2" rowspan="5">$\alpha\delta\epsilon$</td><td>$\beta\delta^2, \beta^2\epsilon$</td></tr>
<tr><td>$\beta\delta^2, \gamma\epsilon^2, \alpha^3$</td></tr>
<tr><td>$\beta\delta^2, \gamma^2\epsilon$</td></tr>
<tr><td>$\beta\delta^2, \gamma^3$</td></tr>
<tr><td>$\beta\delta^2, \epsilon^3$</td></tr>
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<table>
<tr><th colspan="3">Necessary</th><th>Optional</th></tr>
<tr><td rowspan="19">$\alpha\beta\gamma$</td><td rowspan="12">$\alpha\delta^2$</td><td rowspan="3">$\alpha^2\epsilon$</td><td>$\beta\epsilon^2$</td></tr>
<tr><td>$\beta^2\delta$</td></tr>
<tr><td>β^3</td></tr>
<tr><td rowspan="3">$\beta\epsilon^2$</td><td>$\alpha^2\epsilon$</td></tr>
<tr><td>$\gamma^2\delta$</td></tr>
<tr><td>γ^3</td></tr>
<tr><td rowspan="3">$\beta^2\epsilon$</td><td>$\gamma\epsilon^2$</td></tr>
<tr><td>$\gamma^2\delta$</td></tr>
<tr><td>γ^3</td></tr>
<tr><td rowspan="2">$\delta\epsilon^2$</td><td>$\beta^2\epsilon$</td></tr>
<tr><td>β^3</td></tr>
<tr><td>ϵ^3</td><td>$\beta^2\delta$</td></tr>
<tr><td rowspan="6">$\alpha^2\delta$</td><td rowspan="3">$\beta^2\epsilon$</td><td>$\alpha\epsilon^2$</td></tr>
<tr><td>$\gamma\delta^2$</td></tr>
<tr><td>γ^3</td></tr>
<tr><td rowspan="2">$\delta^2\epsilon$</td><td>$\beta^2\epsilon$</td></tr>
<tr><td>β^3</td></tr>
<tr><td>ϵ^3</td><td>$\beta\delta^2$</td></tr>
<tr><td colspan="2">$\delta\epsilon^2$</td><td>α^3</td></tr>
<tr><td colspan="2" rowspan="3">$\alpha\beta^2, \gamma\delta^2$</td><td rowspan="2">$\alpha^2\epsilon$</td><td>$\beta\gamma^2$</td></tr>
<tr><td>$\delta\epsilon^2$</td></tr>
<tr><td>ϵ^3</td><td>$\alpha^2\delta$</td></tr>
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Table 1: Anglewise vertex combinations at degree 3 vertices.

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